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BIFURCATION DIAGRAM OF A MAP WITH MULTIPLE CRITICAL POINTS

M. ROMERA[1], G. PASTOR[1], M.-F. DANCA[2], A. MARTIN[1], A.B. ORUE[1], F. MONTOYA[1], L. HERNÁNDEZ ENCINAS[1], E. ŢUNDREA[3]

[1] *Instituto de Tecnologías Físicas y de la Información (ITEFI)*
Consejo Superior de Investigaciones Científicas (CSIC)
Serrano 144, Madrid, 28006, Spain
{*miguel, gerardo, agustin, amalia.orue, fausto, luis*}*@iec.csic.es*

[2] *Romanian Institute of Science and Technology,*
Cireşilor 29, Cluj-Napoca, 400487, România.
danca@rist.ro

[3] *Emanuel University of Oradea,*
Nufărului 87, 410597, Oradea, România.
emanuel@emanuel.ro

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In this work a conjecture to draw the bifurcation diagram of a map with multiple critical points is enunciated. The conjecture is checked by using two quartic maps in order to verify that the bifurcation diagrams obtained according to the conjecture contain all the periodic orbits previously counted by Xie and Hao for maps with four laps.
We show that a map with split bifurcation contains more periodic orbits than those counted by these authors for a map with the same number of laps.

Keywords: bifurcation diagram; nonlinear discrete dynamical system; quartic map; critical point; hidden bifurcation diagrams.

1. Introduction

The bifurcation diagram of a map with one critical point was studied for the first time by May ([May, 1976]). He used the logistic difference equation $x_{n+1} = ax_n(1 - x_n)$, and gave a catalogue of its stable cycles. Starting from the cubic-difference equation $x_{n+1} = ax_n^3 + (1-a)x_n$, Testa and Held ([Testa & Held, 1983]) showed that the bifurcation diagram of this map, with two critical points, exhibits a split bifurcation not found in maps with one critical point. As a result, the final states of the system depend on the initial conditions, and the number of periodic orbits is different to the number of periodic orbits of the logistic map. Jánosi and Gallas ([Jánosi & Gallas, 1999]) found that the bifurcation diagram of the quartic map $f(x) = 1 - a(1 - ax^2)^2$, with three critical points, preserves the basic bifurcation structure of the logistic map in parameter space; however, in sharp contrast with the logistic case, it displays two coexisting stable attractors.

In previous works, we have studied quartic maps and bifurcation diagrams of maps with more than a critical point ([Danca *et al.*, 2009], [Danca *et al.*, 2013], [Romera *et al.*, 2015], and [Pastor *et al.*, 2016]). As a further step, in current paper we introduce a conjecture to draw the bifurcation diagram of any map with multiple critical points.

By using this conjecture, we have drawn the bifurcation diagrams of two quartic maps which have three critical points and four laps. On the other hand, Xie & Hao ([Xie & Hao, 1994]) and Hao ([Hao, 2000]) already counted the number of periodic orbits in maps with multiple critical points. Thus, we can see if the number of periodic orbits of our two quartic maps with three critical points and four laps coincides with the number of orbits given by Xie & Hao and Hao for the same type of maps.

There are two scenarios, bifurcation diagrams with and without split bifurcation. As we shall see, in the cases without split bifurcation, the number of periodic orbits given by Xie & Hao and Hao coincides with the number of periodic orbits of our bifurcation diagrams. However, in the cases with split bifurcation, we find a number of periodic orbits greater than the given by Xie & Hao and Hao for the same number of laps.

2. Conjecture

In order to draw the bifurcation diagram of any map with multiple critical points we introduce the following

Conjecture

The bifurcation diagram of a map with multiple critical points is the set of bifurcation diagrams corresponding to each one of the critical values of the map.

Xie and Hao ([Xie & Hao, 1994]) and Hao ([Hao, 2000]) have counted the number of periodic orbits in maps with multiple critical points and they have also shown that a map with 4 laps has 2 period-1 orbits, 4 period-2 orbits, 10 period-3 orbits, 32 period-4 orbits, and 102 period-5 orbits. This fact allows us to verify the conjecture in the quartic maps $x_{n+1} = 1 - a(1 - ax_n^2)^2$ and $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$, by computing the values of the parameter a that cause orbits of periods $1, 2, 3, 4$ and 5 and checking that these orbits and their corresponding bifurcation diagrams are inside the bifurcation diagram defined by the conjecture.

3. The quartic map $x_{n+1} = 1 - a(1 - ax_n^2)^2$

Let us consider the second iteration of the quadratic map $x_{n+1} = 1 - ax_n^2$, i.e. $x_{n+2} = 1 - a(1 - ax_n^2)^2$. Obviously, the quartic map $x_{n+1} = 1 - a(1 - ax_n^2)^2$ coincides with the just seen second iteration of the quadratic map $x_{n+1} = 1 - ax_n^2$.

The quartic map $x_{n+1} = 1 - a(1 - ax_n^2)^2$, has three critical points ($0, \sqrt{1/a}$ and $-\sqrt{1/a}$), four laps, and two critical values ($1 - a$ and 1) (Fig. 1). Obviously, the critical points $\sqrt{1/a}$ and $-\sqrt{1/a}$ are real when $a > 0$, complex when $a < 0$ and tends to infinity when a tends to 0. We will only study this map when $a > 0$.

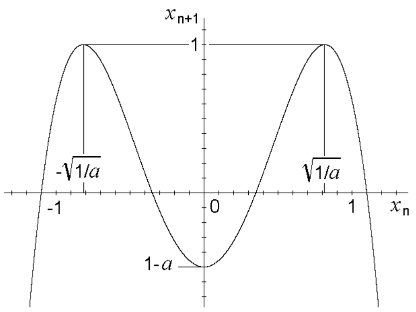


Fig. 1. Critical points and critical values of the quartic map $x_{n+1} = 1 - a(1 - ax_n^2)^2$.

4. Bifurcation diagram of the map $x_{n+1} = 1 - a(1 - ax_n^2)^2$

According to the Conjecture, the bifurcation diagram of this map is the set of the two bifurcation diagrams obtained when the initial points are the critical values $1 - a$ and 1 (Fig. 2).

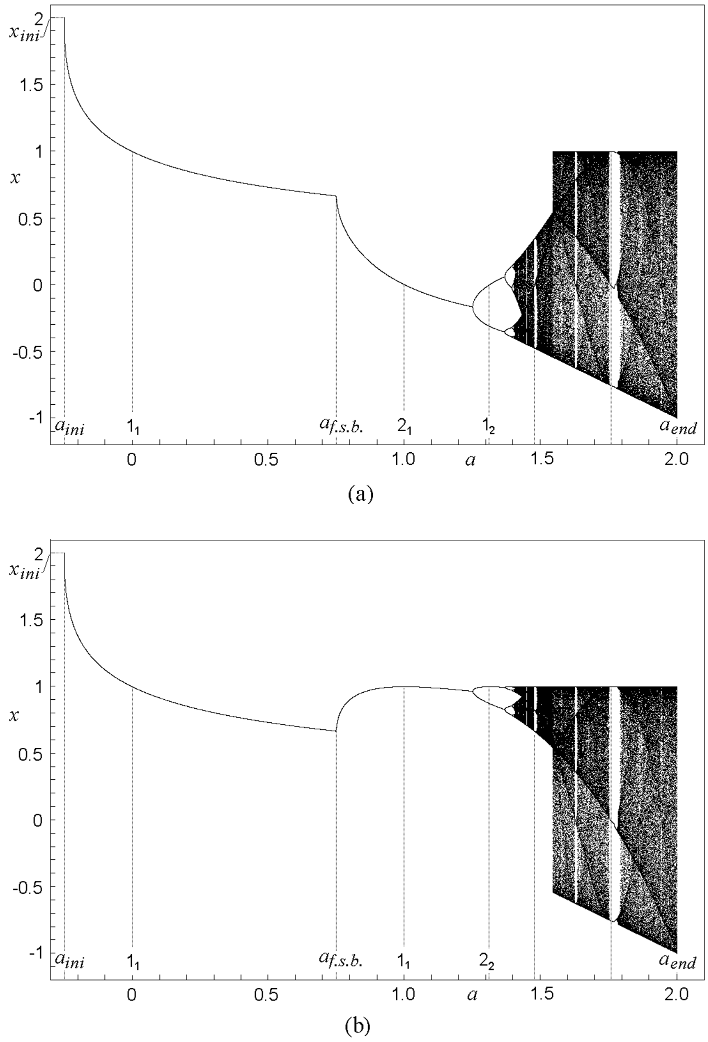


Fig. 2. Bifurcation diagram of the map $x_{n+1} = 1 - a(1 - ax_n^2)^2$. (a) Component corresponding to the critical value $1 - a$. (b) Component corresponding to the critical value 1.

4.1. *Initiation of the bifurcation diagram*

It is easy to see that the initiation of the bifurcation diagram, a_{ini} in Fig. 2, happens when the graph of the map is tangent to the straight line $x_{n+1} = x_n$ in a point x_{ini}, that is

$$4a_{ini}^2 x_{ini}(1 - a_{ini}x_{ini}^2) = 1. \tag{1}$$

Since x_{ini} belongs to $x_{n+1} = x_n$, then $1 - a_{ini}(1 - a_{ini}x_{ini}^2)^2 = x_{ini}$ and, therefore,

$$a_{ini}(1 - a_{ini}x_{ini}^2)^2 = 1 - x_{ini} \tag{2}$$

Dividing member by member Eq. (1) by Eq. (2) one obtains

$$x_{ini} = \frac{2\,a_{ini} \pm \sqrt{4\,a_{ini}^2 - 3\,a_{ini}}}{3\,a_{ini}} \tag{3}$$

Substituting Eq. (3) into Eq. (1) one gets $a_{ini} = -0.25$ and, finally, $x_{ini} = 2$.

4.2. *First split bifurcation*

The map $x_{n+1} = 1 - a(1 - ax_n^2)^2$ presents the split bifurcation. As is known, according to [Testa & Held, 1983], a split bifurcation qualitatively resembles to a pitchfork bifurcation with half of the pitchfork missing.

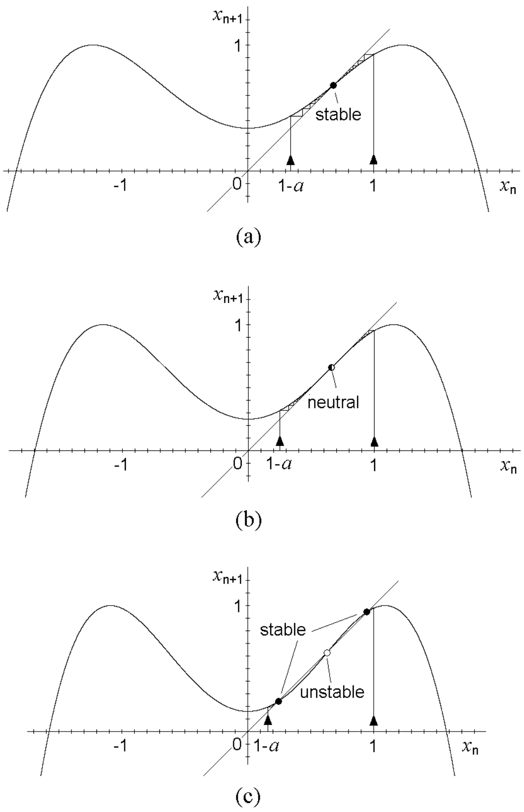


Fig. 3. First split bifurcation of the map $x_{n+1} = 1 - a(1 - ax_n^2)^2$. (a) $a = 0.66$. (b) $a = a_{f.s.b.} = 0.75$. (c) $a = 0.84$.

In Fig. 3 we analyze the first split bifurcation of the map $x_{n+1} = 1 - a(1 - ax_n^2)^2$ that starts, as calculated later, in $a_{f.s.b.} = 0.75$ (see Fig. 2). Shortly before this value, for instance when $a = 0.66$, the map has a stable fixed point (with slope lower than 1) that can be reached by iteration from the critical value $1 - a$ and also from the critical value 1. When $a = 0.75$, the map is tangent to the straight line $x_{n+1} = x_n$ in a neutral inflection point (slope equal to 1), that is also reached by iteration from the critical values $1-a$ and 1. Analytically, we have $f(x) = 1 - a(1-ax^2)^2$, $f'(x) = 4\,a^2 x(1-ax^2)$, $f''(x) = 4\,a^2(1-3\,ax^2)$, and the inflection point $\left(\sqrt{1/3\,a},\ 1-4\,a/9\right)$. Therefore, $\sqrt{1/3\,a} = 1-4\,a/9$ and $a = 0.75$. Shortly after this value, for instance when $a = 0.84$, the fixed point is unstable (slope greater than 1) and in its proximity two stable fixed points appear; one of them is reached by iteration from the critical value $1-a$ and the other one is reached from the critical value 1.

As will be seen later, the split bifurcation also appears when the graph of the period-p iteration of the map is tangent to the line $x_{n+p} = x_n$ in p inflection points, and what has just been described above also occurs in each one of these inflection points.

4.3. *Parameter values of superstable period-1, 2, . . . 5 orbits*

The parameter values of superstable period orbits are obtained by iterating the map $x_{n+1} = 1-a(1-ax_n^2)^2$, first with the initial point $x_0 = 1-a$ and then with the initial point $x_0 = 1$. In each case, the values of the parameter a corresponding to the period-i superstable orbits ($i = 1, 2, \ldots 5$) are the solutions of the equations in a that are obtained from $x_{n+1} = 1 - a(1-ax_n^2)^2$ making $x_i = x_0$.

The results are summarized in Tables 1 and 2.

Table 1. Equations for the parameter values of superstable orbits with initial point $x_0 = 1-a$ for the map $x_{n+1} = 1 - a(1-ax_n^2)^2$.

Period	Equation	
1	$a - a\left(1-a(1-a)^2\right)^2 = 0$	(4)
2	$a - a\left(1-a\left(1-a\left(1-a(1-a)^2\right)^2\right)^2\right)^2 = 0$	(5)
i (i=3,4,5)	$a - a\left(\overbrace{1-a\cdots\left(1-a\right.}^{2i}\ \overbrace{\left.\right)^2\cdots}^{2i}\right)^2 = 0$	(6)

Table 2. Equations for the parameter values of superstable orbits with initial point $x_0 = 1$ for the map $x_{n+1} = 1 - a(1-ax_n^2)^2$.

Period	Equation	
1	$a(1-a)^2 = 0$	(7)
2	$a\left(1-a\left(1-a(1-a)^2\right)^2\right)^2 = 0$	(8)
i (i=3,4,5)	$a\left(\overbrace{1-a\cdots\left(1-a\right.}^{2i-1}\ \overbrace{\left.\right)^2\cdots}^{2i-1}\right)^2 = 0$	(9)

4.4. *Number of period-1 orbits*

The number of period-1 orbits is calculated by means of Eq. (4) and Eq. (7). Eq. (4) has the solutions $a = 0$ (fixed point $x = 1$), $a = 1$ (fixed point $x = 0$), and $a = 2$ (see Fig. 2(a)). This later solution is an unstable fixed point and should be neglected. However, it is the end of the bifurcation diagram $a_{end} = 2$. As can be remarked in Fig. 4(a), when $a = 2$, the graphical iteration from the critical values $1 - a$ and 1 leads to the unstable fixed point M and, when $a > 2$, this iteration goes to the infinity (Fig. 4(b)).

Eq. (7) has the solutions $a = 0$ (fixed point $x = 1$) and $a = 1$ (fixed point $x = 1$) as can be seen in Fig. 2(b). Note that in the a-axis of Fig. 2(b) there are two 1_1, one in $a = 0$ and another one in $a = 1$. This can also be observed in Table 3.

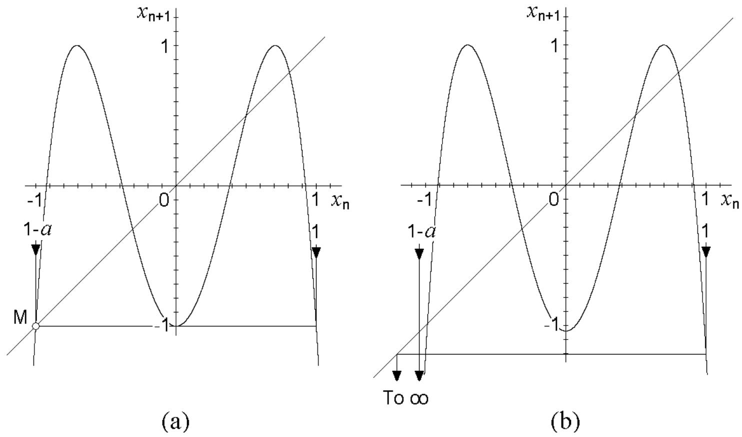


Fig. 4. Graphical iteration of the map $x_{n+1} = 1 - a(1 - ax_n^2)^2$ from the critical values $1 - a$ and 1. (a) $a = 2$. (b) $a = 2.04$.

As a result, in the quartic map $x_{n+1} = 1 - a(1 - ax_n^2)^2$ there are two period-1 orbits in the bifurcation diagram of Fig.2, (see Table 3), what is in accordance with Table IV of Xie and Hao ([Xie & Hao, 1994]) and Table 1 of Hao ([Hao, 2000]) for maps with four laps. However, let us note the obvious first split bifurcation that begins in $a_{f.s.b.} = 0.75$.

Table 3. Period-1 orbits of the map $x_{n+1} = 1 - a(1 - ax_n^2)^2$.

Number	a	Initial	Orbit	Observations	Figure
	0	$1 - a$	Fixed point 1		Fig. 2(a)
1_1	0	1	Fixed point 1		Fig. 2(b)
	1	1	Fixed point 1	Half of the pitchfork bifurcation	Fig. 2(b)
2_1	1	$1 - a$	Fixed point 0	Half of the pitchfork bifurcation	Fig. 2(a)

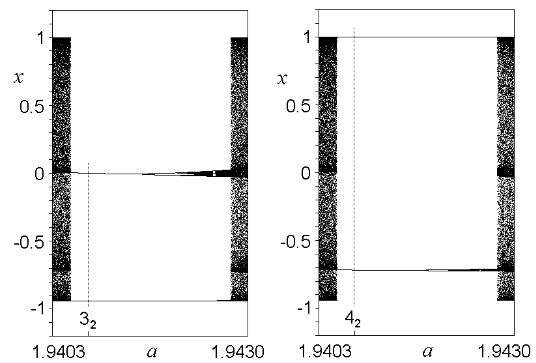
4.5. *Number of period-2 orbits*

The number of period-2 orbits is calculated by means of Eq. (5) and Eq. (8). Neglecting the solutions $a = 0$ and $a = 2$ (corresponding to fixed points), we have the solutions showed in Table 4.

As a result, in the quartic map $x_{n+1} = 1 - a(1 - ax_n^2)^2$ there are 4 period-2 orbits in the bifurcation diagram of Fig. 2, what is in accordance with Table IV of Xie and Hao ([Xie & Hao, 1994]) and Table 1 of Hao ([Hao, 2000]).

Table 4. Period-2 orbits of the map $x_{n+1} = 1 - a(1 - ax_n^2)^2$.

Number	a	Initial	Orbit	Figure
1_2	1.310702	$1-a$	$-0.310702,\ 0$	Fig. 2(a)
2_2	1.310702	1	1, 0.873470	Fig. 2(b)
3_2	1.940799	$1-a$	$-0.940799,\ 0$	Fig. 5(a)
4_2	1.940799	1	$1,\ -0.717810$	Fig. 5(b)



(a) (b)

Fig. 5. Some period-2 windows in the map $x_{n+1} = 1 - a(1 - ax_n^2)^2$. (a) Magnification of the a axis of Fig. 2(a). (b) Magnification of the a axis of Fig. 2(b).

4.6. *Number of period-3 orbits*

The number of period-3 orbits is calculated by means of Eq. (6) and Eq. (9) making $i = 3$. Neglecting the solutions $a = 0$ and $a = 1$ (corresponding to fixed points), we have the valid solutions of Table 5.

Table 5. Period-3 orbits of the map $x_{n+1} = 1 - a(1 - ax_n^2)^2$.

Number	a	Initial	Orbit	Observations	Figure
1_3	1.476014	$1 - a$	$-0.476014,\ 0.346187,\ 0$		Fig. 6(a)
2_3	1.476014	1	$1,\ -0.665551,\ 0.823104$		Fig. 6(f)
3_3	1.754877	$1 - a$	$-0754877,\ 1,\ 0$		Fig. 6(b)
	1.754877	1	$1,\ 0,\ -0754877$		Fig. 6(g)
4_3	1.772892903	$1 - a$	$-0.772892903,\ 0.993815707,\ 0$	Half of the pitchfork bifurcation	Fig. 6(b)
5_3	1.772892903	1	$1,\ -0,059061402.\ -0.751032532$	Half of the pitchfork bifurcation	Fig. 6(g)
6_3	1.907280	$1 - a$	$-0.907280,\ 0.380344,\ 0$		Fig. 6(c)
7_3	1.907280	1	$1,\ -0.569990,\ 0.724090$		Fig. 6(h)
8_3	1.966773	$1 - a$	$-0.966773,\ -0.381960,\ 0$		Fig. 6(d)
9_3	1.966773	1	$1,\ -0.838244,\ 0.713054$		Fig. 6(i)
10_3	1.996376	$1 - a$	$-0.996376,\ -0.924888,\ 0$		Fig. 6(e)
11_3	1.996376	1	$1,\ -0.981932,\ -0.707748$		Fig. 6(j)

As can be seen from Table 5, the quartic map $x_{n+1} = 1 - a(1 - ax_n^2)^2$ has 11 period-3 orbits, instead of the 10 period-3 orbits of Table IV of Xie and Hao ([Xie & Hao, 1994]) and Table 1 of Hao ([Hao, 2000]) both for maps with four laps. Our new results (in Table 5) are due to the existence of one split bifurcation that is not considered in the mentioned works.

The period-3 orbits of Table 5 are shown in Fig. 6. In Fig. 6(a) and Fig. 6(f) one can see that there exists a period doubling bifurcation cascades. Also, Fig. 6(b) and Fig. 6(g) reveal a split bifurcation with the orbits 3_3 and 4_3 (Fig. 6(b)) and also with the orbits 3_3 and 5_3 (Fig. 6(g)).

Figs. 7(a) and 7(b) are, respectively, magnifications of the x axis of Fig. 6(b) and Fig. 6(g) to better show the split bifurcation that begins when $a = 1.768550$ with the x values -0.763, -0.029 and 0.999.

In Fig. 8 we can see the graph of the third iteration of the map $x_{n+1} = 1 - a(1 - ax_n^2)^2$, i.e. $x_{n+3} = 1 - a\left(1 - a\left(1 - a\left(1 - a\left(1 - a(1 - ax_n^2)^2\right)^2\right)^2\right)^2\right)^2$, for the parameter value $a = 1.768550$. Note that it has three inflection points located on the straight line $x_{n+3} = x_n$ in the x values -0.763, -0.029 and 0.999, according to Fig. 7.

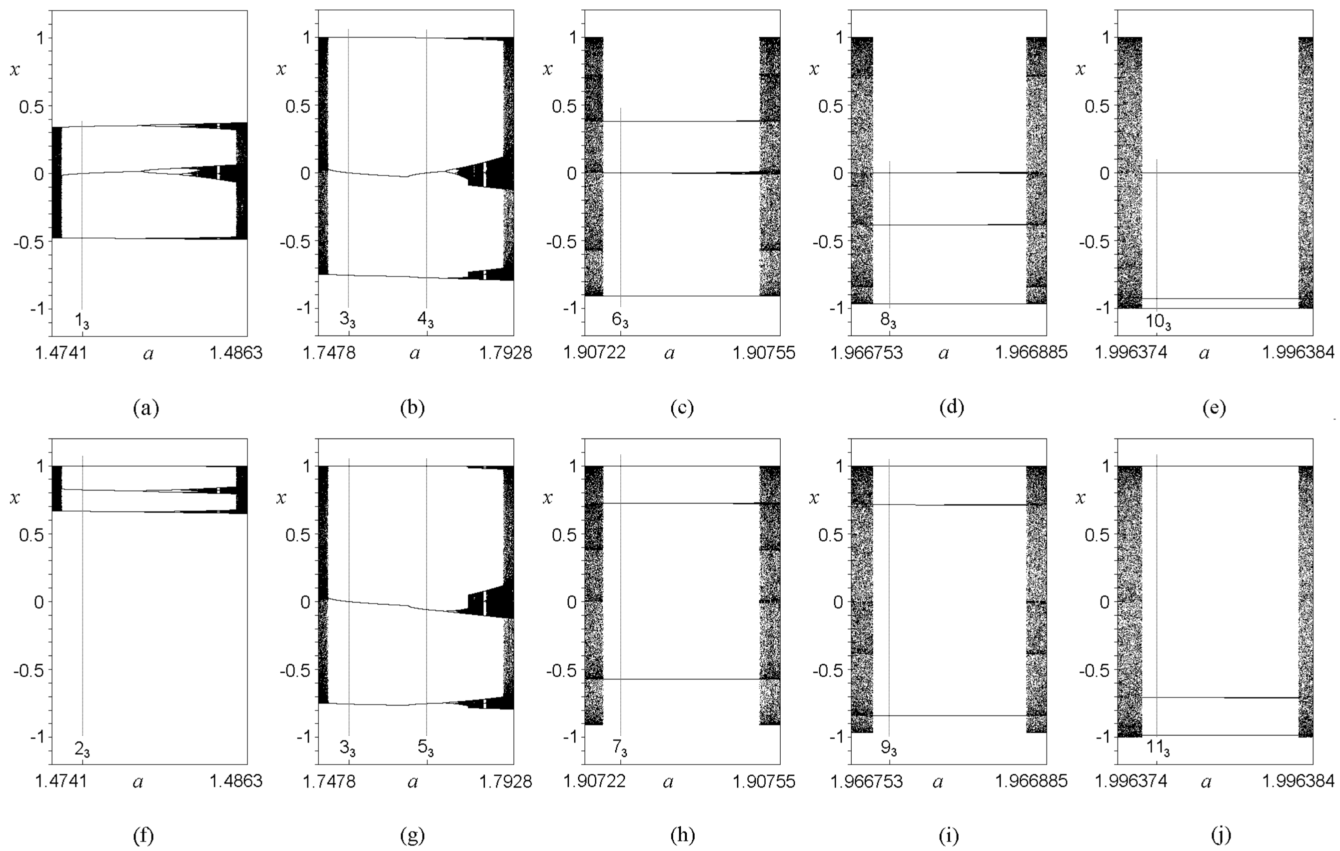


Fig. 6. Period-3 windows of the map $x_{n+1} = 1 - a(1 - ax_n^2)^2$. (a) - (e): magnification of the a axis of Fig. 2(a). (f) - (j): magnification of the a axis of Fig. 2(b).

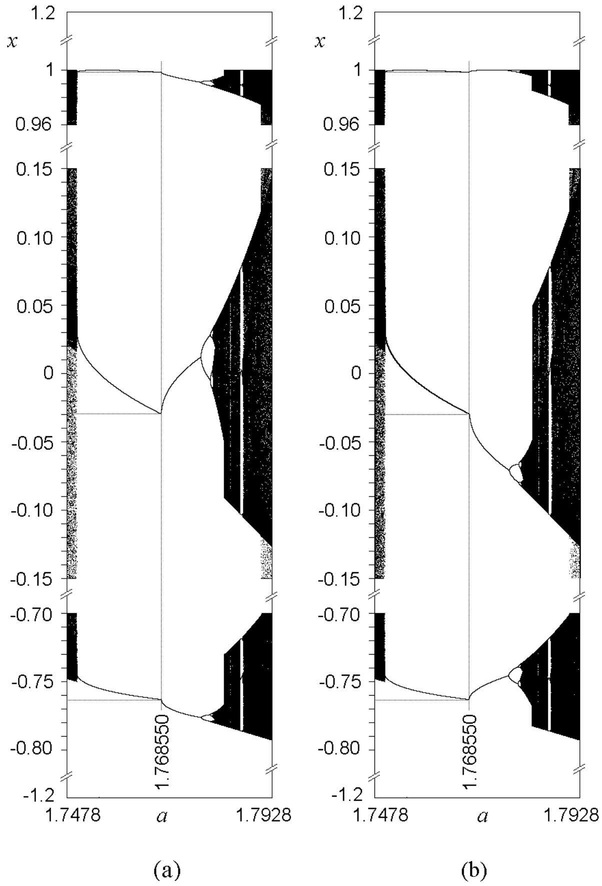


Fig. 7. Map $x_{n+1} = 1 - a(1 - ax_n^2)^2$. Detail of the split bifurcation in a period-3 window. (a) Magnification of the x axis of Fig. 6(b). (b) Magnification of the x axis of Fig. 6(g).

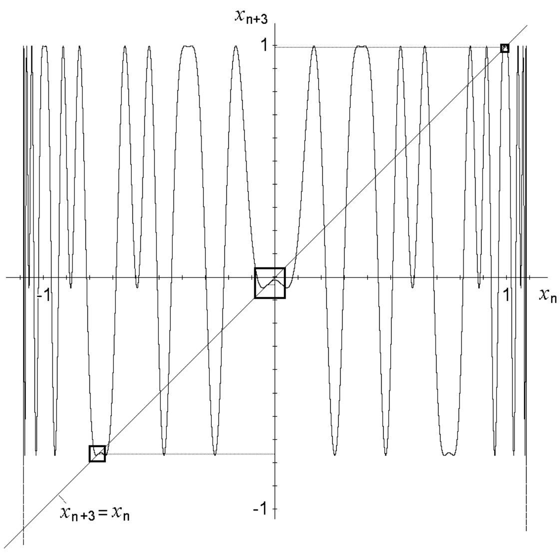


Fig. 8. Map $x_{n+1} = 1 - a(1 - ax_n^2)^2$. Graph of the third iteration for $a = 1.768550$.

4.7. *Number of period-4 orbits*

The number of period-4 orbits can be calculated by means of Eq. (6) and Eq. (9) making $i = 4$. Neglecting the solutions $a = 0$ and $a = 1$ (corresponding to fixed points), and $a = 1.940799$ (period-2 orbit), we have 16 solutions that originate 32 orbits. Some of them are shown in Table 6.

Table 6. Some of the period-4 orbits of the map $x_{n+1} = 1 - a(1 - ax_n^2)^2$.

Number	a	Initial	Orbit
1_4	1.381547	$1 - a$	$-0.381547,\ 0.118290,\ -0.328648,\ 0$
2_4	1.381547	1	$1,\ 0.798877,\ 0.980668,\ 0.850779$
...	...	...	...
9_4	1.851730	$1 - a$	$-0.851730,\ 0.781731,\ 0.967931,\ 0$
10_4	1.851730	1	$1,\ -0.343326,\ -0.131598,\ -0.734869$
...	...	...	...
15_4	1.917098	$1 - a$	$-0.917098,\ 0.280997,\ -0.380632,\ 0$
16_4	1.917098	1	$1,\ -0.612411,\ 0.848627,\ 0.722249$
...	...	...	...
23_4	1.981655	$1 - a$	$-0.981655,\ -0.639620,\ 0.929005,\ 0$
24_4	1.981655	1	$1,\ -0.909615\ 0.189277,\ -0.710269$
...	...	...	...
31_4	1.999774	$1 - a$	$-0.999774,\ -0.995257,\ -0.923926,\ 0$
32_4	1.999774	1	$1,\ -0.998870,\ -0.980852,\ -0.707086$

As a result, in the quartic map $x_{n+1} = 1 - a(1 - ax_n^2)^2$ there are 32 period-4 orbits in the bifurcation diagram of Fig. 2, what is in accordance with Table IV of Xie and Hao ([Xie & Hao, 1994]) and Table 1 of Hao ([Hao, 2000]) for maps with four laps.

4.8. *Number of period-5 orbits*

The number of period-5 orbits is calculated by means of Eq. (6) and Eq. (9) making $i = 5$. Neglecting the solutions $a = 0$, $a = 1$, and $a = 2$ (corresponding to fixed points), we obtain the solutions of Table 7. Both Eq. (6) and Eq. (9), with $i = 5$, have 54 solutions that originate $2 \times 54 - 3 = 105$ orbits because there are three split bifurcations. Some of these orbits are shown in Table 7.

As a result, in the quartic map $x_{n+1} = 1 - a(1 - ax_n^2)^2$ there are 105 period-5 orbits instead of the 102 period-5 orbits of Table IV of Xie and Hao ([Xie & Hao, 1994]) and Table 1 of Hao ([Hao, 2000]).

4.9. *Summary of the analysis of* $x_{n+1} = 1 - a(1 - ax_n^2)^2$

Summarizing the results of the analysis of $x_{n+1} = 1 - a(1 - ax_n^2)^2$, three cases can be considered. In the first case, for period-2 and period-4 orbits, the number of orbits given by us coincide with the number given by Table IV of Xie and Hao ([Xie & Hao, 1994]) and Table 1 of Hao ([Hao, 2000]). This is because for period-2 and period-4 orbits there are no split bifurcation.

In the second case, for period-3 and period-5 orbits, we have found a number of orbits greater that the number given in the just mentioned tables. For period-3 orbits, we find one more orbit because there

Table 7. Some of the period-5 orbits of the map $x_{n+1} = 1 - a(1 - ax_n^2)^2$.

Number	a	Initial	Orbit	Observations
1_5	1.447008	$1 - a$	$-0.447008,\ 0.268786,\ -0.160280,\ -0.341427,\ 0$	
2_5	1.447008	1	$1,\ 0.710864,\ 0.895459,\ 0.962826,\ 0.831318$	
...	...	...	...	...
7_5	1.625413 1.625413	$1 - a$ 1	$-0.625413,\ 0.784362,\ 1,\ 0.364233,\ 0$ $1,\ 0.364233,\ 0,\ -625413,\ 0.784362$	
8_5	1.629432	$1 - a$	$-0.629432,\ 0.795293,\ 0.998474,\ 0.364596,\ 0$	Half of the pitchfork bifurcation
9_5	1.629432	1	$1,\ 0.354444,\ -0.030602,\ -0.624462,\ 0.783398$	Half of the pitchfork bifurcation
...	...	...	...	...
24_5	1.860782 1.860782	$1 - a$ 1	$-0.860782,\ 0.733084,\ 1,\ -0.378738,\ 0$ $1,\ -0.378738,\ 0,\ -0.860782,\ 0.733084$	
25_5	1.861558	$1 - a$	$1,\ -0.861558,\ 0.728636,\ 0.999746,\ -0.378771,\ 0$	Half of the pitchfork bifurcation
26_5	1.861558	1	$1,\ -0.381801,\ 0.011678,\ -0.860612,\ 0.732926$	Half of the pitchfork bifurcation
...	...	...	...	...
73_5	1.985424 1.985424	$1 - a$ 1	$-0.985424,\ -0.709692,\ 1,\ -0.927953,\ -0$ $1,\ -0.927966,\ 0,\ -0.985424,\ -0.709692$	
74_5	1.985482	$1 - a$	$-0.985482,\ -0.710786,\ -0.000017,\ -0.927953,\ 0$	Half of the pitchfork bifurcation
75_5	1.985482	1	$1,\ -0.928250,\ 0,\ -0.985406,\ -0.709693$	Half of the pitchfork bifurcation
...	...	...	...	...
104_5	1.999986	$1 - a$	$-0.999986,\ -0.999706,\ -0.995229,\ -0.924516,\ 0$	
105_5	1.999986	1	$1,\ -0.999930,\ -0.998810,\ -0.980950,\ -0.709447$	

is one split bifurcation, and for period-5 orbits we find three more orbits because there are three split bifurcations.

In the third case, for period-1 orbit, although there is one split bifurcation, the number of orbits given by us coincide with the number given by Table IV of Xie and Hao ([Xie & Hao, 1994]) and Table 1 of Hao ([Hao, 2000]). This is explained in section 4.4 where we can see that for $a = 0$ the map $x_{n+1} = 1-a(1-ax_n^2)^2$ is reduced to $x_{n+1} = 1$ and, consequently, as shown in Fig. 2 and Table 3, there are only two fixed points in total.

5. Map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ for a given value of b

For a given value of b, the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ has three critical points: 0 (critical value $1 - a$), $\sqrt{1/b}$ (critical value 1), and $-\sqrt{1/b}$ (critical value 1). In accordance with the Conjecture, the bifurcation diagram of this map is the set of bifurcation diagrams obtained by taking as initial points the critical values $1 - a$, Fig. 9(a), and 1, Fig. 9(b).

The values of the parameter a that originate period-i orbits ($i = 1, 2, 3, 4, 5$) taking as initial point $x_0 = 1 - a$ are obtained as follows: first, by iteration, we obtain x_i ($i = 1, 2, 3, 4, 5$); next we consider $x_i = 1 - a$. In this way we obtain the results in Table 8.

Table 8. Equations for the parameter values of periodic orbits with initial point $x_0 = 1-a$ for the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ with a given b.

Period	Equation	
1	$a - a\left(1 - b(1-a)^2\right)^2 = 0$	(10)
2	$a - a\left(1 - b\left(1 - a\left(1 - b(1-a)^2\right)^2\right)^2\right)^2 = 0$	(11)
i (i=3,4,5)	$a - a\left(\overbrace{1 - b\left(1 - a \cdots \left(1 - b\left(1 - a\right.\right.\right.}^{i} \overbrace{\left.\right)^2 \cdots \right)}^{2i}\right)^2 = 0$	(12)

Similarly, the values of the parameter a that originate period-i orbits ($i = 1, 2, 3, 4, 5$) by taking as initial point $x_0 = 1$ are obtained as follows: first, by iteration we obtain x_i ($i = 1, 2, 3, 4, 5$); next consider $x_i = 1$. In this case we obtain the results in Table 9.

Table 9. Equations for the parameter values of periodic orbits with initial point $x_0 = 1$ for the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ for a given b.

Period	Equation	
1	$a(1-b)^2 = 0$	(13)
2	$a\left(1 - b\left(1 - a(1-b)^2\right)^2\right)^2 = 0$	(14)
i (i=3,4,5)	$a\left(\overbrace{1 - b\left(1 - a \cdots \left(1 - b\left(1 - a\right.\right.\right.}^{i-1}\ (1 - b \overbrace{\left.\right)^2 \cdots \right)}^{2i-1}\right)^2 = 0$	(15)

Let us note that Fig. 9(b) has two different parts: a standard bifurcation diagram when $a_{ini} < a < a_{end(s)}$ and a hidden bifurcation diagram ([Pastor *et al.*, 2016]) when $a_{end(s)} < a < a_{end(h)}$. Inside the hidden bifurcation diagram, the periodic windows are very narrow, being between orbits that go to infinity, and hence, are very difficult to detect. The standard part was drawn quickly and with good resolution using a sweep of the parameter $\triangle a = 2 \times 10^{-4}$, while the hidden part was drawn, very slowly and with very poor resolution, using a sweep of $\triangle a = 10^{-9}$.

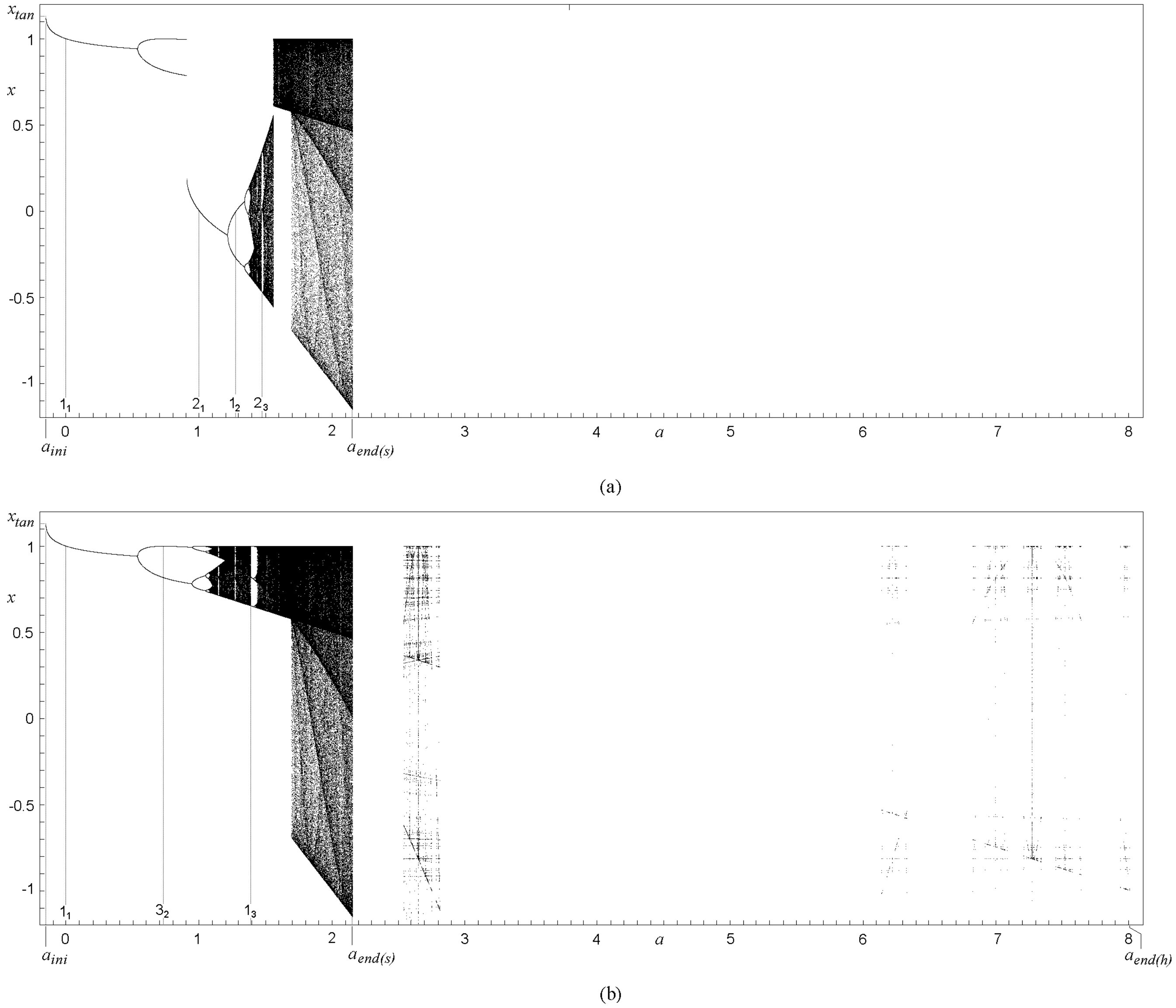


Fig. 9. Bifurcation diagram of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$. (a) Component corresponding to the critical value $1 - a$. (b) Component corresponding to the critical value 1.

5.1. *Number of superstable period-1 orbits*

The parameter values of period-1 orbits of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$ can be calculated by means of Eq. (10) and Eq. (13) with $b = 1.5$.

Eq. (10) has the solutions $a = -0.154700$ (neutral period-1 orbit and initial point of the bifurcation diagram a_{ini}), $a = 0$ (superstable fixed point 1), $a = 1$ (superstable fixed point 0), and $a = 2.154700$ (unstable period-1 orbit and the end of standard bifurcation diagram $a_{end(s)} = 2.154700$, Fig. 10).

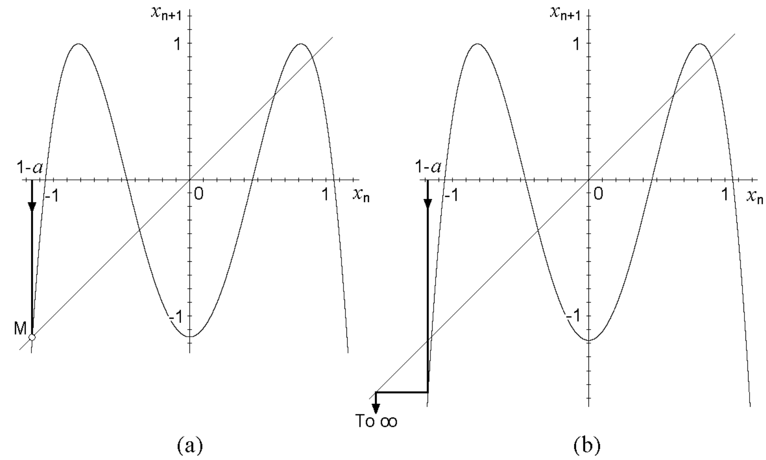


Fig. 10. Graphical iteration of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$, starting from the critical value $1 - a$. (a) If $a = a_{end(s)} = 2.154700$ an unstable fixed point M is obtained. (b) If $a > a_{end(s)}$ the orbit always goes to infinity.

Eq. (13) has only the solution $a = 0$ (superstable fixed point 1). Let us note that $a_{end(s)}$ in Fig. 9(b) obviously coincides with $a_{end(s)}$ in Fig. 9(a) but $a = 2.154700$ is not a solution of period-1 orbit in Eq. (13). This is because the iteration starting from the critical value 1 goes to the same point M of Fig. 10 but now M is a Misiurewicz point with preperiod-3 and period-1, as we can see in Fig. 11(a). For this reason, when $a > a_{end(s)}$ and it is inside the hidden part of the bifurcation diagram, the orbit can go to infinity, Fig. 11(b), or can be periodic, Fig. 11(c).

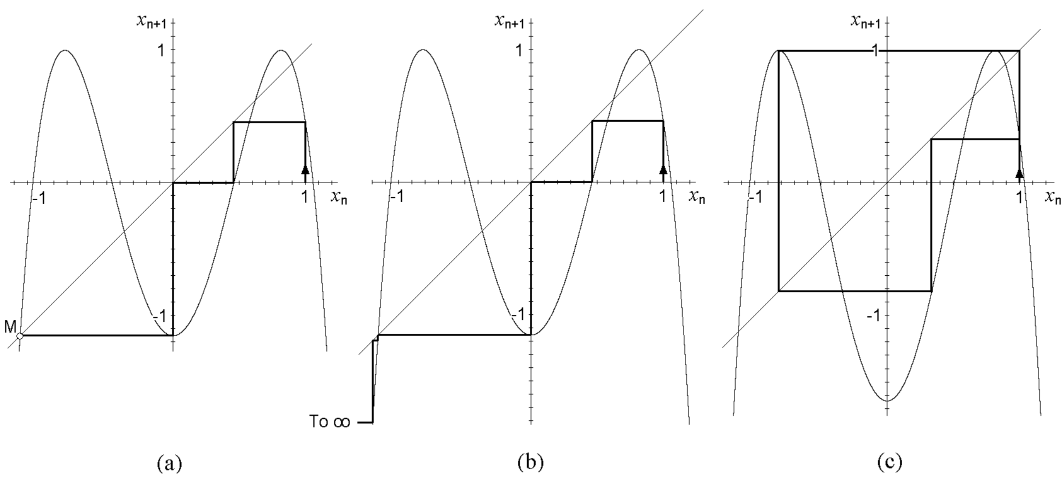


Fig. 11. Graphical iteration of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$, starting from the critical value 1. (a) $a = a_{end(s)} = 2.154700$. (b) $a = 2.155000$, orbit to infinity. (c) $a = 2.647048$, period-3 orbit.

In this way, the superstable period-1 orbits of the map $x_{n+1} = 1 - a(1 - bx^2)^2$, when $b = 1.5$, are shown in Table 10.

Table 10. Superstable period-1 orbits of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$.

Number	a	Initial	Orbit	Figure
1_1	0	$1 - a$	Fixed point 1	Fig. 9(a)
		1	Fixed point 1	Fig. 9(b)
2_1	1	$1 - a$	Fixed point 0	Fig. 9(a)

As a result, in the quartic map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$ there are 2 period-1 orbits in the bifurcation diagram of Fig. 9, what is in accordance with Table IV of Xie and Hao ([Xie & Hao, 1994]) and Table 1 of Hao ([Hao, 2000]).

5.2. *Number of superstable period-2 orbits*

The parameter values of period-2 orbits of map when $b = 1.5$ are calculated by means of Eq. (11) and Eq. (14) with $b = 1.5$.

Neglecting the period-1 solutions, $a = -0.154700$, $a = 0$, and $a = 1$, Eq. (11) has the solutions $a = 1.276969$ and a=2.063378. Neglecting the period-1 solution $a = 0$, Eq. (14) has the solutions $a = 0.734013$ and $a = 7.265986$. In this way, the superstable period-2 orbits of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$ are shown in Table 11.

Table 11. Superstable period-2 orbits of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$.

Number	a	Initial	Orbit	Figure
1_2	1.276969	$1 - a$	-0.276969, 0	Fig. 9(a)
2_2	2.063378	$1 - a$	-1.063378. 0	Fig. 12(a)
3_2	0.734013	1	1, 0.816496	Fig. 9(b)
4_2	7.265986	1	1, -0.816496	Fig. 12(b)

In Fig. 12 one can see the bifurcation diagrams of two period-2 orbits: number 2_2 in the standard diagram of Fig. 9(a), and 4_2 in the hidden diagram of Fig. 9(b). Note that on the left and on the right side of these bifurcation diagrams, in the first case the orbits are finite whereas in the second case the orbits go to infinity.

Note that the orbit number 3_2 of Table 11 and Fig. 9(b) was only obtained starting from the critical value 1 according to Eq. (14). However, it also appears in Fig. 9(a) because it was asymptotically achieved (not directly) as we see in Fig. 13. For this reason 3_2 was not obtained by Eq. (11).

As a result, in the quartic map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$ there are 4 period-2 orbits in the bifurcation diagram of Fig. 9, what is in accordance with Table IV of Xie and Hao ([Xie & Hao, 1994]) and Table 1 of Hao ([Hao, 2000]).

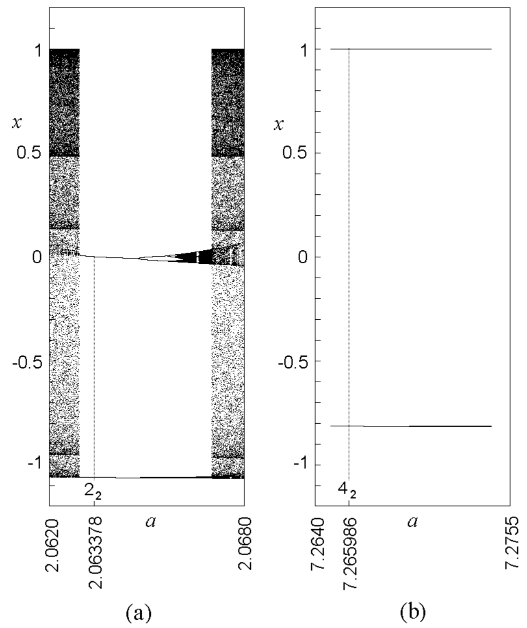


Fig. 12. Two magnifications of Fig. 9 showing period-2 windows of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$. (a) Magnification from Fig. 9(a). (b) Magnification from Fig. 9(b).

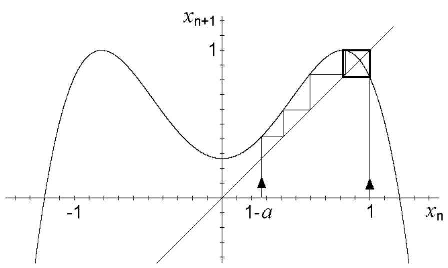


Fig. 13. Graphical iteration of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$ and $a = 0.734013$, starting from the critical values $1 - a$ and 1.

5.3. *Number of superstable period-3 orbits*

In order to obtain the parameter values of period-3 orbits of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$, one can use Eq. (12) and Eq. (15) with $i = 3$ and $b = 1.5$.

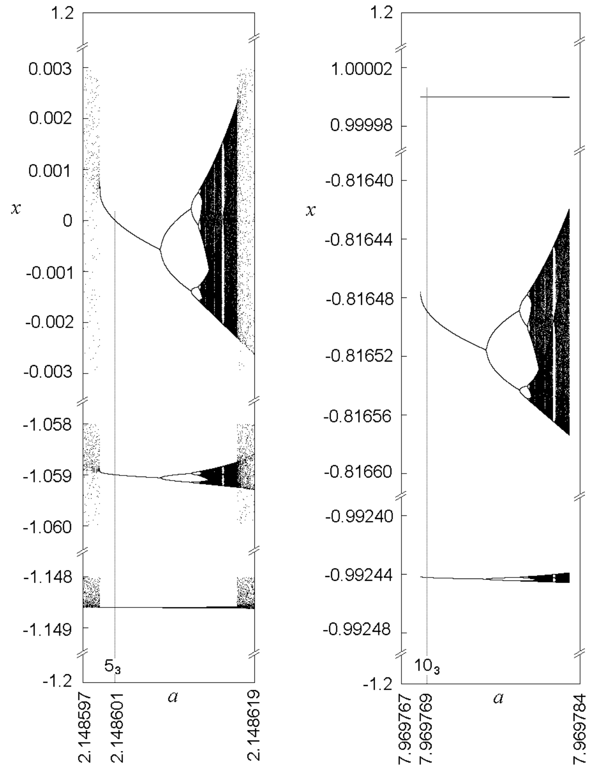
Neglecting the period-1 solutions of Eq. (12), $a = -0.154701$, $a = 0$, $a = 1$ and $a = 2.154700$ (all of them already considered) and the period-1 solution $a = 0$ of Eq. (15), the superstable period-3 orbits are given in Table 12.

Table 12. Superstable period-3 orbits of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ for $b = 1.5$.

Number	a	Initial	Orbit	Figure
1_3	1.393188	1	$1,\ 0.651702,\ 0.816496$	Fig. 9(b)
2_3	1.469512	$1-a$	$-0.469512,\ 0.341639,\ 0$	Fig. 9(a)
3_3	2.008810	$1-a$	$-1.008810,\ 0.443055,\ 0$	
4_3	2.104971	$1-a$	$-1.104971,\ -0.455155,\ 0$	
5_3	2.148601	$1-a$	$-1.148601,\ -1.058997,\ 0$	Fig. 14(a)
6_3	2.647048	1	$1,\ 0.338238,\ -0.816496$	
7_3	6.213427	1	$1,\ -0.553356,\ -0.816496$	
8_3	6.989708	1	$1,\ -0.747427,\ 0.816496$	
9_3	7.511953	1	$1,\ -0.877988,\ 0.816496$	
10_3	7.969769	1	$1,\ -0.992442,\ -0.816496$	Fig. 14(b)

As a result, in the quartic map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$ there are 10 period-3 orbits in the bifurcation diagram of Fig. 9, in accordance with Table IV of Xie and Hao ([Xie & Hao, 1994]) and Table 1 of Hao ([Hao, 2000]).

In Fig. 14 one can see the bifurcation diagrams of period-3 orbits 5_3 and 10_3 (Table 12). They are obtained by extending a and x axes in the bifurcation diagrams of Fig. 9(a) and Fig. 9(b), respectively.



(a) (b)

Fig. 14. Two of the ten period-3 windows of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$. (a) Window of the orbit 5_3. (b) Window of the orbit 10_3.

5.4. *Number of superstable period-4 orbits*

The parameter values of period-4 orbits of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$ are calculated by means of Eq. (12) and Eq. (15) with $i = 4$ and $b = 1.5$. The trivial solutions of Eq. (12) $a = -0.154701$, $a = 0$ and $a = 1$ (period-1 orbits), and $a = 1.276969$ and $a = 2.063378$ (period-2 orbits) have been ignored as the trivial solutions of Eq. (15) $a = 0$ (period-1 orbit), and $a = 0.734013$ and 7.265986 (period-2 orbits). Some superstable period-4 orbits are shown in Table 13.

Table 13. Superstable period-4 orbits of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$.

Number	a	Initial	Orbit	Figure
1_4	0.998045	1	$1, 0.750488, 0.975975, 0.816496$	Fig. 9(b)
2_4	1.355856	$1-a$	$-0, 355856,\ 0.110313,\ -0.306809,\ 0$	Fig. 9(a)
...	...	...	...	
9_4	2.094798	$1-a$	$-1.094798,\ -0.333554,\ -0.453948,\ 0$	
10_4	2.114330	$1-a$	$-1.114330,\ -0.573217,\ 0.456228,\ 0$	
...	...	...	...	
13_4	2.154292	$1-a$	$-1.154292,\ -1.148206,\ -1.058714,\ 0$	Fig. 15(a)
14_4	2.336039	1	$1,\ 0.415990,\ -0.280696,\ -0.816498$	
...	...	...	...	
19_4	6.186371	1	$1,\ -0.546593,\ -0.884019,\ 0.816485$	
20_4	6.242787	1	$1,\ -0.560697,\ -0.743217, 0.816506$	
...	...	...	...	
25_4	7.270185	1	$1,\ -0.817546,\ 0.999952,\ -0.816497$	Fig. 15(b)
26_4	7.468297	1	$1,\ -867074,\ 0.878162,\ 0.816496$	
...	...	...	...	
31_4	7.979806	1	$1,\ -0.994951,\ -0.876220,\ 0.816504$	
32_4	7.998742	1	$1,\ -0.999686,\ -0.992147,\ -0.816388$	

As a result, in the quartic map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$ there are 32 period-4 orbits in the bifurcation diagram of Fig. 9, in accordance with Table IV of Xie and Hao ([Xie & Hao, 1994]) and Table 1 of Hao ([Hao, 2000]).

In Fig. 15 one can see the bifurcation diagrams of period-4 orbits 13_4 and 25_4 (Table 13). They are obtained by extending a and x axes in the bifurcation diagrams in Fig. 9(a) and Fig. 9(b), respectively.

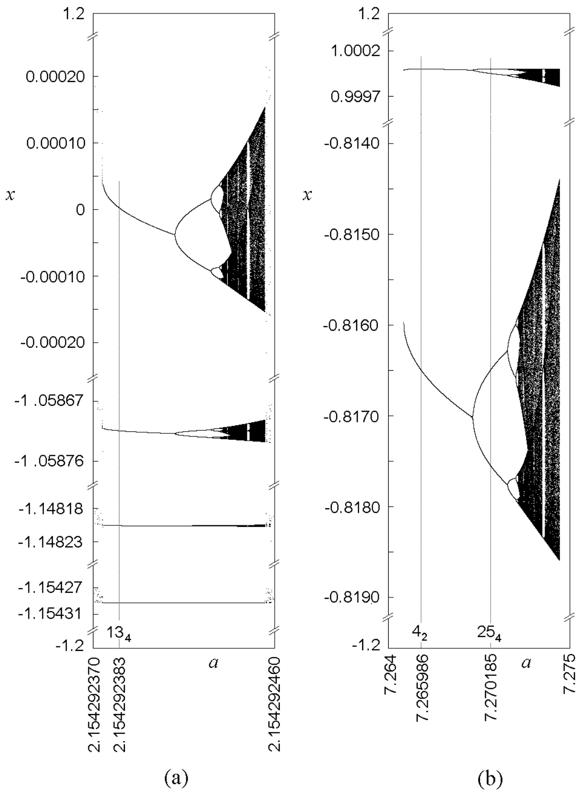


Fig. 15. Map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ for $b = 1.5$. (a) Period-4 orbit 13_4 in a period-4 window. (b) Period-4 orbit 25_4 in a period-2 window.

5.5. *Number of superstable period-5 orbits*

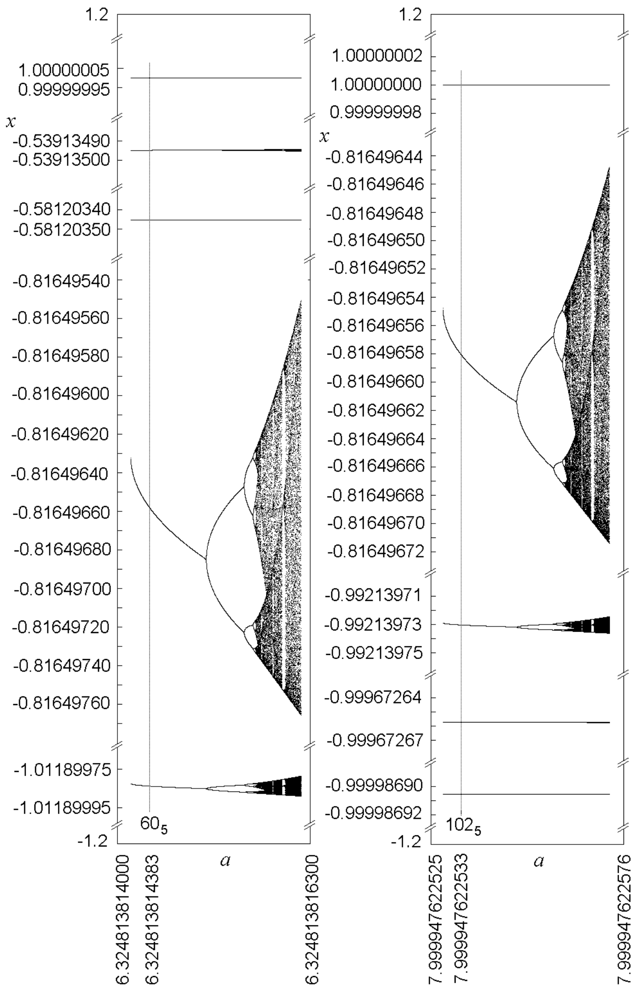
The parameter values of period-5 orbits of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ with $b = 1.5$ are calculated by means of Eq. (12) and Eq. (15) making $i = 5$. One ignore the trivial solutions of Eq. (12) $a = -0.154701$, $a = 0$ and $a = 1$ (period-1 orbits). Also the trivial solution of Eq. (15) $a = 0$ (period-1 orbit) is ignored. Some of the superstable period-5 orbits are shown in Table 14.

Table 14. Some period-5 orbits of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ with $b = 1.5$.

Number	a	Initial	Figure
1_5	1.266106204944	1	
2_5	1.433632715233	$1 - a$	
...	...	...	
22_5	2.116858295007	$1 - a$	
23_5	2.136856683482	1	
...	...	...	
40_5	2.685191489972	1	
41_5	2.716369883125	1	
60_5	6.324813814383	1	Fig.16(a)
61_5	6.816582379086	1	
...	...	...	
80_5	7.543751023347	1	
81_5	7.556786579882	1	
...	...	...	
101_5	7.999159621025	1	
102_5	7.999947622533	1	Fig.16(b)

As a result, in the quartic map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ when $b = 1.5$ there are 102 period-5 orbits in the bifurcation diagram of Fig. 9, what is in accordance with Table IV of Xie and Hao ([Xie & Hao, 1994]) and Table 1 of Hao ([Hao, 2000]).

In Fig. 16 one can see the bifurcation diagrams of the orbits 60_5 and 102_5 with the parameter values given in Table 14.



(a) (b)

Fig. 16. Magnifications of Fig. 9(b) showing the bifurcation diagrams of two period-5 orbits of the map $x_{n+1} = 1-a(1-bx_n^2)^2$ with $b = 1.5$. (a) Orbit 60_5. (b) Orbit 102_5.

5.6. *Summary of the analysis of* $x_{n+1} = 1 - a(1 - bx_n^2)^2$ *for a given value of* b

If we summarize the results of the analysis of the map $x_{n+1} = 1 - a(1 - bx_n^2)^2$ for a given b, in all the cases the number of orbits given by us coincides with the number given by Xie and Hao in Table IV of [Xie & Hao, 1994] and by Hao in Table 1 of [Hao, 2000]. This is because in no one of the cases there exists any split bifurcation.

6. Conclusions

In this paper we have enunciated a conjecture to draw the bifurcation diagram of a map with multiple critical points. The conjecture has been verified using two maps with three critical points. In this way, we have been able to verify the existence of the periodic orbits counted by Xie and Hao ([Xie & Hao, 1994]) and Hao ([Hao, 2000]), until period-5, for maps with four laps. Those orbits appear in the drawn diagrams.

One of the important findings of this paper is the fact that one of the two used maps exhibits split bifurcation and its bifurcation diagram contains more periodic orbits than those counted by the previous authors. The question of knowing if a map with multiple critical points exhibits split bifurcation is an open problem.

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